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The electromagnetic interaction of a massive spin one particle: some equivalence theorems and remarks

J D JENKINS

Department of Physics, University of Durham, South Road, Durham, UK

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Abstract. The minimal electromagnetic interaction is introduced into various free, quantum mechanical spin one formalisms, and it is found that the resulting theories are essentially equivalent. This situation does not prevail when, in addition, magnetic dipole or electric quadrupole interactions are considered. For simple such interactions in one formalism are found, in general, to be equivalent to structurally more complicated interactions in the other formalisms considered. Some known results on the acausal propagation of solutions of wave equations for a classical spin one particle in an external electromagnetic field are, with the aid of the above equivalence theorems, which are easily seen to apply to the classical case also, extended to include all the spin one formalisms considered herein.

1. Introduction

In past years, in addition to the commonly used vector field, various other equivalent quantum mechanical formalisms for a free massive spin one particle have been proposed. Amongst these are those due to Proca (1936), Duffin and Kemmer (Duffin 1938, Kemmer 1939), Stueckelberg (1938), Hammer *et al* (1968), Takahashi and Palmer (1970) and Macfarlane and Tait (1972). The last three formalisms are closely related; for that of Takahashi and Palmer is just that of Hammer *et al* written in tensor, rather than spinor form, whilst that of Macfarlane and Tait is the dual of the Takahashi–Palmer formalism, and has also been considered, in a slightly different manner, by Jenkins (1972a).

In contrast with the *a priori* equivalence of the above free spin one formalisms, the construction of equivalent theories of an interacting spin one particle, in different formalisms, is, as illustrated by the examples of Jenkins (1972a, 1972b), not, in general, so simple. In this paper the relationships between different spin one formalisms are explored further by a consideration of the electromagnetic interactions of a spin one particle, described by any one of the above formalisms. Some work in this direction has already been done by Bludman and Young (1963). They demonstrated that, in the presence of the minimal electromagnetic interaction, the vector, Proca, Duffin–Kemmer and Stueckelberg formalisms are essentially equivalent. This result is here extended to include the formalisms of Hammer *et al*, Takahashi and Palmer and Macfarlane and Tait; and equivalence theorems for anomalous magnetic dipole and electric quadrupole interactions, in the various spin one formalisms, are also established. Some remarks on the equivalence theorems are made, and an application is considered. The plan of the paper is as follows.

In § 2 the minimal electromagnetic interaction is introduced into the Proca formalism, and, following Jenkins (1972b), the equivalent theories in terms of the vector and anti-symmetric second rank tensor spin one fields are constructed. A first order formalism, analogous to the Proca formalism, and falling naturally between the vector formalism and a formalism simply related to that of Takahashi and Palmer, is constructed in § 3. The relationship of this last formalism to the Takahashi–Palmer formalism is clarified. Then the minimal electromagnetic interaction is introduced into the first order formalism, and the equivalent theories in terms of the vector and Takahashi–Palmer spin one fields are constructed. In § 4 equivalence theorems are established for the formalisms considered in §§ 2 and 3 when, in addition to the minimal electromagnetic interaction, anomalous magnetic dipole and electric quadrupole interactions are included. In § 5, an ambiguity, first noted by Bludman and Young (1963), involved in the introduction of the minimal electromagnetic interaction into free spin-one Lagrangians is discussed; and its implications for the equivalence theorems of §§ 2, 3 and 4 are considered. On noting that the form of the equivalence theorems, in §§ 2, 3 and 4, remains valid in the case of a classical spin one particle in an external electromagnetic field, they are used, in § 6, in conjunction with some known results of Velo and Zwanziger (1969) and Shamaly and Capri (1972), concerning the possible acausal propagation of, respectively, classical vector and Takahashi–Palmer spin one fields in an external electromagnetic field, to make some comparative remarks and extend their results to the other formalisms considered in this paper. Section 7 is devoted to a discussion of the results of this paper, and their implications. Finally it should be noted that, throughout this paper, all quantum mechanical considerations will be in terms of the Heisenberg picture.

2. The Proca formalism

The free Lagrangian for a nonhermitian Proca field

$$\chi(x) = \begin{pmatrix} V_\mu(x) \\ \phi_{\alpha\beta}(x) \end{pmatrix}$$

is given by

$$\mathscr{L}(x) = \mu[V^{\dagger\mu}(x)\phi^{\dagger\alpha\beta}(x)]\begin{pmatrix} \mu g_{\mu\nu} & \frac{1}{\sqrt{2}}(g_{\rho\mu}\partial_\lambda - g_{\lambda\mu}\partial_\rho) \\ \frac{1}{\sqrt{2}}(g_{\beta\nu}\bar{\partial}_\alpha - g_{\alpha\nu}\bar{\partial}_\beta) & \mu l_{\alpha\beta\lambda\rho} \end{pmatrix}\begin{pmatrix} V^\nu(x) \\ \phi^{\lambda\rho}(x) \end{pmatrix} \tag{1}$$

where the field $\phi_{\alpha\beta}(x)$ is antisymmetric in its indices, and $l_{\alpha\beta\lambda\rho} = \frac{1}{2}(g_{\alpha\lambda}g_{\beta\rho} - g_{\alpha\rho}g_{\beta\lambda})$. The minimal electromagnetic interaction is now introduced by the usual gauge-invariant prescription $\partial_\mu\phi_{\alpha\beta}(x) \to \pi_\mu\phi_{\alpha\beta}(x) = (\partial_\mu + \mathrm{i}eA_\mu(x))\phi_{\alpha\beta}(x)$ etc, where $A_\mu(x)$ is the electromagnetic potential and e the electric charge of the spin one particle. The resulting Lagrangian is

$$\mathscr{L}(x) = \mu[V^{\dagger\mu}(x)\phi^{\dagger\alpha\beta}(x)]\begin{pmatrix} \mu g_{\mu\nu} & \frac{1}{\sqrt{2}}(g_{\rho\mu}\pi_\lambda - g_{\lambda\mu}\pi_\rho) \\ \frac{1}{\sqrt{2}}(g_{\beta\nu}\bar{\pi}^\dagger_\alpha - g_{\alpha\nu}\bar{\pi}^\dagger_\beta) & \mu l_{\alpha\beta\lambda\rho} \end{pmatrix}\begin{pmatrix} V^\nu(x) \\ \phi^{\lambda\rho}(x) \end{pmatrix} \tag{2}$$
$$-\tfrac{1}{2}\partial^\mu A^\nu(x)\partial_\mu A_\nu(x)$$

and the corresponding equations of motion are

$$\mu^2 V_\mu(x) + \frac{\mu}{\sqrt{2}}(\pi^\lambda \phi_{\lambda\mu}(x) - \pi^\lambda \phi_{\mu\lambda}(x)) = 0 \tag{3}$$

$$\mu^2 \phi_{\mu\nu}(x) - \frac{\mu}{\sqrt{2}}(\pi_\mu V_\nu(x) - \pi_\nu V_\mu(x)) = 0 \tag{4}$$

$$\partial^2 A_\mu(x) = J_\mu(x) \equiv -ie\mu\sqrt{2}(V^{\dagger\nu}(x)\phi_{\mu\nu}(x) - \phi^\dagger_{\mu\nu}(x)V^\nu(x)) \tag{5}$$

where $J_\mu(x)$ is just the gauge-invariant current density vector for the Proca field.

Now, following Jenkins (1972b), the theories of the electromagnetic interaction of a spin one particle, described by a vector or an antisymmetric second rank tensor field, which are equivalent to the above Proca theory, are constructed.

Firstly the Lagrangian, equivalent to (2), in which the action of all the derivatives appearing in (2) has been transferred, by the introduction of appropriate divergence terms, from $\phi_{\alpha\beta}(x)$ onto $V_\mu(x)$ is considered. Equation (4) is now used to eliminate $\phi_{\alpha\beta}(x)$ from this Lagrangian and equations (3) and (5). The resulting Lagrangian and equations of motion are

$$\mathscr{L}(x) = -\tfrac{1}{2}(\pi_\mu V_\nu(x) - \pi_\nu V_\mu(x))^\dagger(\pi^\mu V^\nu(x) - \pi^\nu V^\mu(x)) + \mu^2 V^\dagger_\mu(x)V^\mu(x) - \tfrac{1}{2}\partial^\mu A^\nu(x)\partial_\mu A_\nu(x) \tag{6}$$

$$\pi^\mu(\pi_\mu V_\nu(x) - \pi_\nu V_\mu(x)) + \mu^2 V_\nu(x) = 0 \tag{7}$$

$$\partial^2 A_\mu(x) = K_\mu(x) \equiv -ie(V^{\dagger\nu}(x)(\pi_\mu V_\nu(x) - \pi_\nu V_\mu(x)) - (\pi_\mu V_\nu(x) - \pi_\nu V_\mu(x))^\dagger V^\nu(x)) \tag{8}$$

where, as is easily checked, $K_\mu(x)$ is just the gauge-invariant current density vector for the spin one vector field. The inverse transformation is effected by use of (4), regarded as the definition of the antisymmetric tensor part of the Proca field in terms of $V_\mu(x)$. On noting that (6), (7), (8) are just the Lagrangian and equations of motion for the minimal electromagnetic interaction of the spin one vector field, the well known result (Bludman and Young 1963), that the introduction of the minimal electromagnetic interaction into the free Proca and vector field Lagrangians leads to equivalent theories, is established.

However, in addition, by a use of (3), $V_\mu(x)$ may be eliminated from (2), (4) and (5), giving the theory of the electromagnetic interaction of a $(1, 0) \oplus (0, 1)$ spin one field, which is equivalent to the theory given by (2). The resulting Lagrangian and equations of motion are

$$\mathscr{L}(x) = -\tfrac{1}{2}(\pi^\lambda \phi_{\lambda\mu}(x) - \pi^\lambda \phi_{\mu\lambda}(x))^\dagger(\pi_\rho \phi^{\rho\mu}(x) - \pi_\rho \phi^{\mu\rho}(x)) + \mu^2 \phi^\dagger_{\mu\nu}(x)\phi^{\mu\nu}(x) - \tfrac{1}{2}\partial^\mu A^\nu(x)\partial_\mu A_\nu(x) \tag{9}$$

$$\tfrac{1}{2}(\pi_\mu \pi^\lambda \phi_{\lambda\nu}(x) - \pi_\nu \pi^\lambda \phi_{\lambda\mu}(x) + \pi_\nu \pi^\lambda \phi_{\mu\lambda}(x) - \pi_\mu \pi^\lambda \phi_{\nu\lambda}(x)) + \mu^2 \phi_{\mu\nu}(x) = 0 \tag{10}$$

$$\partial^2 A_\mu(x) = L_\mu(x) \equiv 2ie((\pi_\lambda \phi^{\lambda\rho}(x))^\dagger \phi_{\mu\rho}(x) - \phi^\dagger_{\mu\rho}(x)\pi_\lambda \phi^{\lambda\rho}(x)) \tag{11}$$

where, as is easily checked, $L_\mu(x)$ is just the gauge-invariant current density vector for the $(1, 0) \oplus (0, 1)$ field. The inverse transformation is effected by use of (3), regarded as the definition of the vector part of the Proca field in terms of $\phi_{\alpha\beta}(x)$. Again, (9), (10), (11) are just the Lagrangian and equations of motion for the minimal electromagnetic interaction of the $(1, 0) \oplus (0, 1)$ spin one field considered by Jenkins (1972a).

Collecting results, it has been established that the introduction of the minimal electromagnetic interaction into the free Proca, vector or antisymmetric tensor spin one field Lagrangians leads to theories which are entirely equivalent. These results show that the prescription for introducing the minimal electromagnetic interaction is special, in that it automatically takes into account the contact terms needed for the construction of equivalent theories in terms of the above three spin one fields (Jenkins 1972a, 1972b).

3. The Takahashi–Palmer field

In order to incorporate the spin one theory of Takahashi and Palmer (1970) into the present discussion, a first order formalism, analogous to Proca's, and which is naturally intermediate between the vector and Takahashi–Palmer spin one formalisms, is introduced. The (nonhermitian) field

$$\begin{pmatrix} V_\mu(x) \\ \psi_{\alpha\beta}(x) \end{pmatrix}$$

is assumed to have the free Lagrangian

$$\mathscr{L}(x) = \mu[V^{\dagger\mu}(x)\psi^{\dagger\alpha\beta}(x)] \begin{pmatrix} \mu g_{\mu\nu} & -\frac{1}{\sqrt{2}}\epsilon_{\gamma\mu\lambda\rho}\partial^\gamma \\ -\frac{1}{\sqrt{2}}\epsilon_{\alpha\beta\gamma\nu}\overleftarrow{\partial}^\gamma & -\mu I_{\alpha\beta\lambda\rho} \end{pmatrix} \begin{pmatrix} V^\nu(x) \\ \psi^{\lambda\rho}(x) \end{pmatrix} \tag{12}$$

where $\epsilon_{\mu\nu\lambda\rho}(\epsilon_{0123} = 1)$ is the completely antisymmetric fourth rank tensor, and where $\psi_{\alpha\beta}(x)$ is antisymmetric in its indices. The equations of motion corresponding to (12) are

$$\mu^2 V_\mu(x) - \frac{\mu}{\sqrt{2}}\epsilon_{\gamma\mu\lambda\rho}\partial^\gamma\psi^{\lambda\rho}(x) = 0 \tag{13}$$

$$-\mu^2\psi_{\alpha\beta}(x) + \frac{\mu}{\sqrt{2}}\epsilon_{\alpha\beta\gamma\nu}\partial^\gamma V^\nu(x) = 0. \tag{14}$$

Because of the simple dual relationship between $\phi_{\alpha\beta}(x)$ and $\psi_{\alpha\beta}(x)$, namely

$$\psi_{\alpha\beta}(x) = \tfrac{1}{2}\epsilon_{\alpha\beta\mu\nu}\phi^{\mu\nu}(x) \tag{15}$$

the Klein–Gordon divisor, commutator and free particle propagator in the present formalism are simply obtained by an appropriate insertion of $\epsilon_{\alpha\beta\mu\nu}$ in the corresponding expressions for the Proca formalism (Jenkins 1972b); they are not given here explicitly.

On using the identity

$$\epsilon_{\mu\nu\lambda\rho}\epsilon^{\lambda\rho}{}_{\alpha\beta} = -4I_{\mu\nu\alpha\beta} \tag{16}$$

together with the antisymmetry of $\psi_{\alpha\beta}(x)$, $\psi_{\alpha\beta}(x)$ may be eliminated from the above formalism, and it is readily verified that the resulting Lagrangian and equations of motion are those of a free spin one vector field. If, on the other hand, (13) is used in conjunction with the identity

$$\epsilon_{\mu\nu\lambda}{}^{\rho}\epsilon_{\alpha\beta\gamma\rho} = g_{\mu\alpha}g_{\nu\gamma}g_{\lambda\beta} + g_{\mu\beta}g_{\nu\alpha}g_{\lambda\gamma} + g_{\mu\gamma}g_{\nu\beta}g_{\lambda\alpha} - g_{\mu\alpha}g_{\nu\beta}g_{\lambda\gamma} - g_{\mu\gamma}g_{\nu\alpha}g_{\lambda\beta} - g_{\mu\beta}g_{\nu\gamma}g_{\lambda\alpha}, \tag{17}$$

$V_\mu(x)$ may be eliminated from (12) and (14) to give the following Lagrangian and equations of motion:

$$\mathscr{L}(x) = \tfrac{1}{2}\partial_\lambda\psi^\dagger_{\mu\nu}(x)(\partial^\lambda\psi^{\mu\nu}(x)+\partial^\nu\psi^{\lambda\mu}(x)+\partial^\mu\psi^{\nu\lambda}(x) - \partial^\lambda\psi^{\nu\mu}(x)-\partial^\mu\psi^{\lambda\nu}(x)-\partial^\nu\psi^{\mu\lambda}(x))-\mu^2\psi^\dagger_{\mu\nu}(x)\psi^{\mu\nu}(x) \tag{18}$$

and

$$-(\partial^2+\mu^2)\psi_{\mu\nu}(x)-\partial^\lambda\partial_\mu\psi_{\lambda\nu}(x)+\partial^\lambda\partial_\nu\psi_{\lambda\mu}(x) = 0 \tag{19}$$

where the antisymmetry of $\psi_{\alpha\beta}(x)$ has been used explicitly in obtaining (19). Now (18), apart from some divergence terms, is the Lagrangian of Takahashi and Palmer (1970). However, it should be noted that, whilst they do not assume that $\psi_{\alpha\beta}(x)$ is, *a priori*, antisymmetric, that assumption has been made here. Consequently some remarks on the relationship between the two approaches are in order.

The only essential difference, between the approaches of either requiring the antisymmetry of $\psi_{\alpha\beta}(x)$ to follow from variation of the Lagrangian, or working throughout in the space of antisymmetric second rank tensors, is that, in the former case, the Klein–Gordon divisor contains a term proportional to

$$(\partial^2+\mu^2)(g_{\mu\lambda}g_{\nu\rho}+g_{\mu\rho}g_{\nu\lambda})$$

which is not present in the latter. Evidently, such a term gives zero contribution to the commutator. Hence the only place, where it might have an effect, is in the Green function. However, Takahashi and Palmer make the restriction that if $\psi_{\alpha\beta}(x)$ couples to a source $J_{\alpha\beta}(x)$, say, then $J_{\alpha\beta}(x)$ must be antisymmetric. (Incidentally, this is necessary in order that it be deducible from their formalism, that $\psi_{\alpha\beta}(x)$ is still antisymmetric in the presence of interaction.) And again the above type of term plays no role, since when appropriately contracted with an antisymmetric quantity it gives zero. Thus it is established that it makes no difference whether one uses the Takahashi–Palmer formalism, with the restriction to antisymmetric sources, or restricts oneself, *ab initio*, to the space of antisymmetric second rank tensors.

Collecting together the above remarks, it is seen that, effectively, the formalism given by (12) lies naturally between the vector field and Takahashi–Palmer formalisms, and provides a simple proof of their (known) equivalence in the free-field case.

In the light of the above remarks and manipulations, the introduction of the minimal electromagnetic interaction into (12), and the construction of the equivalent theories in terms of the vector field and the Takahashi–Palmer field is trivial. The details are omitted, and it is merely stated that the vector field Lagrangian and equations of motion are again given by (6), (7) and (8), whilst for the Takahashi–Palmer field

$$\mathscr{L}(x) = \tfrac{1}{2}(\pi_\lambda\psi_{\mu\nu}(x))^\dagger(\pi^\lambda\psi^{\mu\nu}(x)+\pi^\nu\psi^{\lambda\mu}(x)+\pi^\mu\psi^{\nu\lambda}(x)-\pi^\lambda\psi^{\nu\mu}(x)-\pi^\mu\psi^{\lambda\nu}(x)-\pi^\nu\psi^{\mu\lambda}(x)) -\mu^2\psi^\dagger_{\mu\nu}(x)\psi^{\mu\nu}(x)-\tfrac{1}{2}\partial^\mu A^\nu(x)\partial_\mu A_\nu(x) \tag{20}$$

and the corresponding equations of motion are, on an explicit use of the antisymmetry of $\psi_{\alpha\beta}(x)$,

$$-(\pi^2+\mu^2)\psi_{\mu\nu}(x)-\pi^\lambda\pi_\mu\psi_{\lambda\nu}(x)+\pi^\lambda\pi_\nu\psi_{\lambda\mu}(x) = 0 \tag{21}$$

$$\partial^2A_\mu(x) = M_\mu(x) \equiv \frac{ie}{2}(\epsilon^{\alpha\nu\beta\gamma}(\pi_\alpha\psi_{\beta\gamma}(x))^\dagger\epsilon_{\mu\nu\lambda\rho}\psi^{\lambda\rho}(x)-\epsilon_{\mu\nu\lambda\rho}\psi^{\dagger\lambda\rho}(x)\epsilon^{\alpha\nu\beta\gamma}\pi_\alpha\psi_{\beta\gamma}(x)) \tag{22}$$

where, as may be easily checked using (17), $M_\mu(x)$ is just the gauge-invariant current-density vector for the Takahashi–Palmer field. But (20), (21), (22) are just the Lagrangian and equations of motion for the minimal electromagnetic interaction of the Takahashi–Palmer spin one field. Thus it has been established that the introduction of the minimal electromagnetic interaction into the vector field, Takahashi–Palmer and the above first order spin one formalisms leads to theories which are entirely equivalent.

This last result contrasts with a result of Shamaly and Capri (1972) concerning the minimal electromagnetic interaction of the Takahashi–Palmer field. The differences arise as a consequence of an ambiguity inherent in the introduction of the minimal electromagnetic interaction into spin one Lagrangians. This ambiguity, first noted by Bludman and Young (1963), and some of its consequences, are discussed in § 5.

4. Dipole and quadrupole interactions

Some equivalence theorems, concerning dipole and quadrupole interactions of spin one particles, are discussed below in the example of the Proca formalism and its related vector and antisymmetric tensor formalisms. Similar treatments apply to the other formalisms considered in this paper.

The Lagrangian in the Proca formalism, for a spin one particle with anomalous magnetic dipole moment, is taken to be

$$\mathscr{L}(x) = \mu[V^{\dagger\mu}(x)\phi^{\dagger\alpha\beta}(x)]\begin{pmatrix} \mu g_{\mu\nu}+\dfrac{\alpha_1}{\mu}F_{\mu\nu}(x) & -\dfrac{1}{\sqrt{2}}(g_{\rho\mu}\overleftarrow{\pi}^{\dagger}_{\lambda}-g_{\lambda\mu}\overleftarrow{\pi}^{\dagger}_{\rho}) \\ -\dfrac{1}{\sqrt{2}}(g_{\beta\nu}\pi_\alpha-g_{\alpha\nu}\pi_\beta) & \mu l_{\alpha\beta\lambda\rho}+\dfrac{\alpha_2}{\mu}F_{\alpha\beta\lambda\rho}(x) \end{pmatrix}\begin{pmatrix} V^\nu(x) \\ \phi^{\lambda\rho}(x) \end{pmatrix}$$
$$-\tfrac{1}{2}\partial^\mu A^\nu(x)\partial_\mu A_\nu(x) \tag{23}$$

where

$$F_{\alpha\beta\lambda\rho}(x) = \tfrac{1}{2}(g_{\alpha\lambda}F_{\beta\rho}(x)+g_{\beta\rho}F_{\alpha\lambda}(x)-g_{\alpha\rho}F_{\beta\lambda}(x)-g_{\beta\lambda}F_{\alpha\rho}(x)) \tag{24}$$

with

$$F_{\mu\nu}(x) = \partial_\nu A_\mu(x)-\partial_\mu A_\nu(x) \tag{25}$$

being the electromagnetic field-strength tensor. The equations of motion are

$$\mu^2 V_\mu(x)+\frac{\mu}{\sqrt{2}}(\pi^\lambda\phi_{\lambda\mu}(x)-\pi^\lambda\phi_{\mu\lambda}(x))+\alpha_1F_{\mu\lambda}(x)V^\lambda(x) = 0 \tag{26}$$

$$\mu^2\phi_{\mu\nu}(x)-\frac{\mu}{\sqrt{2}}(\pi_\mu V_\nu(x)-\pi_\nu V_\mu(x))+\alpha_2F_{\mu\nu\lambda\rho}(x)\phi^{\lambda\rho}(x) = 0 \tag{27}$$

$$\partial^2A_\mu(x) = J_\mu(x). \tag{28}$$

For simplicity, the case $\alpha_1 \neq 0$, $\alpha_2 = 0$ is first considered. In this case (27) provides a simple relation, between $\phi_{\alpha\beta}(x)$ and $V_\mu(x)$, which allows the immediate elimination of the former from (23), (26) and (28), to give the following Lagrangian and equations of motion:

$$\mathscr{L}(x) = -\tfrac{1}{2}(\pi_\mu V_\nu(x) - \pi_\nu V_\mu(x))^\dagger(\pi^\mu V^\nu(x) - \pi^\nu V^\mu(x)) + \mu^2 V_\mu^\dagger(x)V^\mu(x)$$
$$+\alpha_1 V_\mu^\dagger(x)F^{\mu\nu}(x)V_\nu(x) - \tfrac{1}{2}\partial^\mu A^\nu(x)\partial_\mu A_\nu(x) \tag{29}$$

$$\pi^\mu(\pi_\mu V_\nu(x) - \pi_\nu V_\mu(x)) + \mu^2 V_\nu(x) + \alpha_1 F_{\nu\mu}(x)V^\mu(x) = 0 \tag{30}$$

$$\partial^2 A_\mu(x) = K_\mu(x). \tag{31}$$

But (29), (30) and (31) are just the Lagrangian and equations of motion for the electromagnetic interaction of a spin one vector field with anomalous magnetic dipole moment. On effecting the inverse transformation, in the usual manner, with the aid of (27), the equivalence of the theories given by the Lagrangians (29) and (23), with $\alpha_2 = 0$, is established. However the elimination of $V_\mu(x)$ from the Proca formalism is not such a simple matter, since equation (26) is of the type, mentioned by Jenkins (1972b), which, in general, only allows the elimination to be carried out in iterative fashion. It is evident from the form of (26) that the theory in terms of the antisymmetric tensor field, which is equivalent to those given by the Lagrangians (23) and (29), is given by a Lagrangian which is formally a rational, rather than polynomial, function of the electromagnetic field strengths.

The converse of the above is true in the case $\alpha_1 = 0$, $\alpha_2 \neq 0$. Since the arguments are entirely analogous to those in the above paragraph, with the roles of $\phi_{\alpha\beta}(x)$ and $V_\mu(x)$ interchanged, only the Lagrangian and equations of motion, for the theory in terms of $\phi_{\alpha\beta}(x)$, which is equivalent to that given by (23) with $\alpha_1 = 0$, are written down. They are

$$\mathscr{L}(x) = -\tfrac{1}{2}(\pi^\lambda \phi_{\lambda\mu}(x) - \pi^\lambda \phi_{\mu\lambda}(x))^\dagger(\pi_\rho \phi^{\rho\mu}(x) - \pi_\rho \phi^{\mu\rho}(x)) + \mu^2 \phi_{\mu\nu}^\dagger(x)\phi^{\mu\nu}(x)$$
$$+\alpha_2 \phi_{\alpha\beta}^\dagger(x)F^{\alpha\beta\lambda\rho}(x)\phi_{\lambda\rho}(x) - \tfrac{1}{2}\partial^\mu A^\nu(x)\partial_\mu A_\nu(x) \tag{32}$$

$$\tfrac{1}{2}(\pi_\mu \pi^\lambda \phi_{\lambda\nu}(x) - \pi_\mu \pi^\lambda \phi_{\nu\lambda}(x) + \pi_\nu \pi^\lambda \phi_{\mu\lambda}(x) - \pi_\nu \pi^\lambda \phi_{\lambda\mu}(x)) + \mu^2 \phi_{\mu\nu}(x) + \alpha_2 F_{\mu\nu\lambda\rho}(x)\phi^{\lambda\rho}(x) = 0 \tag{33}$$

$$\partial^2 A_\mu(x) = L_\mu(x) \tag{34}$$

and these are just the Lagrangian and equations of motion for the electromagnetic interaction of a spin one antisymmetric tensor field with anomalous magnetic dipole moment.

Evidently if both α_1, $\alpha_2 \neq 0$, then the vector and antisymmetric tensor spin one Lagrangians, which are equivalent to that given by (23), are both formally given as rational functions of the electromagnetic field strengths.

Thus, in contrast with the case of minimal electromagnetic coupling, there is no simple relationship between equivalent vector and antisymmetric tensor theories of spin one with an anomalous magnetic dipole moment. In addition, the introduction of an anomalous magnetic dipole interaction, in the simplest manner, into vector and antisymmetric tensor spin one theories (namely, with $\alpha_1 \neq 0$, $\alpha_2 = 0$ and $\alpha_1 = 0$, $\alpha_2 \neq 0$ respectively), leads to theories which are inequivalent, in spite of the fact that α_1 and α_2 may be respectively chosen so that the anomalous magnetic dipole moment is the same in each theory. Thus more information, than the value of the anomalous magnetic dipole moment, is needed to distinguish these two simple approaches. This is discussed further in § 6.

Next electric quadrupole coupling is considered. The Lagrangian in the Proca formalism for a spin one particle with anomalous electric quadrupole moment is taken to be

$$\mathscr{L}(x) = \mu[V^{\dagger\mu}(x)\phi^{\dagger\alpha\beta}(x)]\begin{pmatrix} \mu g_{\mu\nu} & \frac{1}{\sqrt{2}}(g_{\rho\mu}\pi_{\lambda}-g_{\lambda\mu}\pi_{\rho}) \\ \frac{1}{\sqrt{2}}(g_{\beta\nu}\bar{\pi}^{\dagger}_{\alpha}-g_{\alpha\nu}\bar{\pi}^{\dagger}_{\beta}) & \mu I_{\alpha\beta\lambda\rho}\end{pmatrix}\begin{pmatrix} V^{\nu}(x) \\ \phi^{\lambda\rho}(x)\end{pmatrix}$$

$$+\frac{\alpha}{\sqrt{2}}(\phi^{\dagger\alpha\beta}(x)Q_{\mu\alpha\beta}(x)V^{\mu}(x)+V^{\dagger\mu}(x)Q_{\mu\alpha\beta}(x)\phi^{\alpha\beta}(x))$$

$$-\tfrac{1}{2}\partial^{\mu}A^{\nu}(x)\partial_{\mu}A_{\nu}(x) \tag{35}$$

where

$$Q_{\mu\alpha\beta}(x) = \partial_{\mu}F_{\alpha\beta}(x).$$

The equations of motion are

$$\mu^2 V_{\mu}(x)+\frac{\mu}{\sqrt{2}}(\pi^{\lambda}\phi_{\lambda\mu}(x)-\pi^{\lambda}\phi_{\mu\lambda}(x))+\frac{\alpha}{\sqrt{2}}Q_{\mu\lambda\rho}(x)\phi^{\lambda\rho}(x) = 0 \tag{36}$$

$$\mu^2\phi_{\mu\nu}(x)-\frac{\mu}{\sqrt{2}}(\pi_{\mu}V_{\nu}(x)-\pi_{\nu}V_{\mu}(x))+\frac{\alpha}{\sqrt{2}}Q_{\lambda\mu\nu}(x)V^{\lambda}(x) = 0 \tag{37}$$

$$\partial^2 A_{\mu}(x) = J_{\mu}(x). \tag{38}$$

Using the methods of the previous sections to construct the equivalent theories in terms of the vector and antisymmetric tensor spin one fields, the corresponding Lagrangians are given respectively by

$$\mathscr{L}(x) = -\tfrac{1}{2}(\pi_{\mu}V_{\nu}(x)-\pi_{\nu}V_{\mu}(x))^{\dagger}(\pi^{\mu}V^{\nu}(x)-\pi^{\nu}V^{\mu}(x))+\mu^2 V^{\dagger}_{\mu}(x)V^{\mu}(x)$$

$$+\frac{\alpha}{2\mu}[(\pi_{\mu}V_{\nu}(x)-\pi_{\nu}V_{\mu}(x))^{\dagger}Q^{\lambda\mu\nu}(x)V_{\lambda}(x)+V^{\dagger}_{\lambda}(x)Q^{\lambda\mu\nu}(x)(\pi_{\mu}V_{\nu}(x)-\pi_{\nu}V_{\mu}(x))]$$

$$-\frac{\alpha^2}{2\mu^2}V^{\dagger}_{\lambda}(x)Q^{\lambda\alpha\beta}(x)Q_{\mu\alpha\beta}(x)V^{\mu}(x)-\tfrac{1}{2}\partial^{\mu}A^{\nu}(x)\partial_{\mu}A_{\nu}(x) \tag{39}$$

and

$$\mathscr{L}(x) = -\tfrac{1}{2}(\pi^{\lambda}\phi_{\lambda\mu}(x)-\pi^{\lambda}\phi_{\mu\lambda}(x))^{\dagger}(\pi_{\rho}\phi^{\rho\mu}(x)-\pi_{\rho}\phi^{\mu\rho}(x))+\mu^2\phi^{\dagger}_{\mu\nu}(x)\phi^{\mu\nu}(x)$$

$$-\frac{\alpha}{2\mu}\{\phi^{\dagger\alpha\beta}(x)Q_{\mu\alpha\beta}(x)(\pi_{\lambda}\phi^{\lambda\mu}(x)-\pi_{\lambda}\phi^{\mu\lambda}(x))+(\pi_{\lambda}\phi^{\lambda\mu}(x)-\pi_{\lambda}\phi^{\mu\lambda}(x))^{\dagger}$$

$$\times Q_{\mu\alpha\beta}(x)\phi^{\alpha\beta}(x)\}-\frac{\alpha^2}{2\mu^2}\phi^{\dagger\alpha\beta}(x)Q_{\mu\alpha\beta}(x)Q^{\mu\lambda\rho}(x)\phi_{\lambda\rho}(x)$$

$$-\tfrac{1}{2}\partial^{\mu}A^{\nu}(x)\partial_{\mu}A_{\nu}(x). \tag{40}$$

Note that the interactions in (39) and (40) both only differ from the simple electric quadrupole interaction by a single contact interaction. However, if a simple electric quadrupole interaction is assumed in the vector formalism, then, although the equivalent interaction for the Proca field will still be simple in form, the construction of the equivalent interaction for the antisymmetric tensor field will be an iterative procedure. The same remarks also apply with the roles of the vector and antisymmetric tensor fields interchanged. To verify these remarks, it is sufficient to note that the unwanted contact interactions in (39) and (40) have the same sort of structure, in terms of the spin one fields, as the previously discussed magnetic dipole interactions, whence the discussion of the latter carries through almost without change.

5. The ambiguity

As noted by Bludman and Young (1963), there are ambiguities inherent in the introduction of the minimal electromagnetic interaction into free spin one particle Lagrangians. They point out that, for example, in the Proca formalism, the addition to the free Lagrangian of a term proportional to

$$\partial_\mu(\partial_\nu V^{\dagger\mu}(x)V^\nu(x)-\partial_\nu V^{\dagger\nu}(x)V^\mu(x)+V^{\dagger\nu}(x)\partial_\nu V^\mu(x)-V^{\dagger\mu}(x)\partial_\nu V^\nu(x))$$

does not affect the free equations of motion, on account of its being a divergence term. However, on the introduction of the minimal electromagnetic interaction, it is no longer a divergence term and so it contributes to the equations of motion, its contribution being a magnetic dipole interaction of the type corresponding to the choice $\alpha_2 = 0$, discussed in § 4.

In a similar manner, the introduction of a divergence term proportional to

$$\partial_\mu(\phi^{\dagger\rho\mu}(x)\partial_\nu\phi_\rho{}^\nu(x)-\phi^{\dagger\rho\nu}(x)\partial_\nu\phi_\rho{}^\mu(x)+\partial_\nu\phi_\rho^{\dagger\nu}(x)\phi^{\rho\mu}(x)-\partial_\nu\phi_\rho^{\dagger\mu}(x)\phi^{\rho\nu}(x))$$

whilst not altering the free equations of motion, leads to a magnetic dipole interaction when minimal electromagnetic coupling is introduced. This time the magnetic dipole interaction is of the type corresponding to the choice $\alpha_1 = 0$, discussed in § 4.

Evidently a magnetic dipole interaction with arbitrary values for α_1 and α_2 may be obtained by introducing the appropriate linear combination of the above two divergence terms into the free Proca Lagrangian.

Thus, although the introduction of magnetic dipole interactions of a charged spin one particle was treated independently of the minimal electromagnetic coupling, in § 4, it may also be considered as arising from the above ambiguity in the introduction of the minimal electromagnetic interaction. However it should be noted that, whilst, as has been seen in § 4, there still exist equivalence theorems for any anomalous magnetic dipole moment, there is only one such value for which all the theories discussed in this paper, with simplest magnetic dipole interactions, are equivalent. This is the value of the intrinsic magnetic dipole moment of the equivalent theories discussed in §§ 2 and 3; and it is given by a gyromagnetic ratio of one (Bludman and Young 1963). For any other value of the gyromagnetic ratio, as evidenced by the results of § 4, the simplest anomalous magnetic dipole interactions, in vector and antisymmetric tensor theories, are not equivalent.

Some comparative remarks, on the theories considered here, may now be made. Firstly, it is noted that the free Takahashi–Palmer equation (19) is just the tensor form of the free spin one equation of Hammer *et al* (1968). However, on the introduction of the minimal electromagnetic interaction the latter equation is symmetric in π_μ (Hammer and Tucker 1971), whilst the former, given by (21), is evidently not. This difference is an example of the above ambiguity, and the two theories differ in the value of the intrinsic magnetic dipole moment. More precisely, in the former case, equivalence to the Proca formalism gives a gyromagnetic ratio of one, whilst in the latter Hammer and Tucker (1971) give its value as one half.

Next, as noted in § 3, the form of the Takahashi–Palmer Lagrangian considered here differs from the original (Takahashi and Palmer 1970) by divergence terms. Thus on the introduction of the minimal electromagnetic interaction the above ambiguity again appears, and these two theories differ in the value of the intrinsic magnetic dipole moment. Finally, the latter theory is also such that the spin one equation is not symmetric in the π_μ, whence it is not equivalent to the minimally coupled theory of Hammer and Tucker (1971), differing in the value of the intrinsic magnetic dipole moment.

Collecting results, it has been seen that the minimally coupled spin one theories of Hammer and Tucker (1971) and Takahashi and Palmer (1970) are not equivalent, and neither is equivalent to the (equivalent) theories considered in §§ 2 and 3. More precisely, in the light of the results of § 4, the minimally coupled Hammer–Tucker and Takahashi–Palmer spin one theories differ from the minimally coupled vector theory by interactions, for the vector field, which are nonpolynomial in the electromagnetic field strengths, and differ from each other and all the other minimally coupled theories, considered in this paper, by simple magnetic dipole interactions.

6. Acausal propagation

Throughout this section, the discussion will be restricted to the classical problem of a spin one particle in an external electromagnetic field. Before proceeding, it should be noted that the essential content of the results of §§ 2, 3, 4 and 5 remain valid in this case, and, incidentally, for a quantized spin one field in an external electromagnetic field.

Velo and Zwanziger (1969) have considered the classical equations for a massive spin one particle, described by a vector field, in the presence of various external fields. They found that, in the presence of certain interactions, the solution of the equations of motion shows acausal propagation, and in some cases the equations of motion cease to be hyperbolic. In particular, they found that a vector field, with arbitrary anomalous magnetic dipole moment, propagates causally. On the other hand, Shamaly and Capri (1972) have given similar consideration to the spin one field of Takahashi and Palmer (1970). They found that, when minimally coupled to an external electromagnetic field, this field propagates acausally; whilst there is just one value of the anomalous magnetic dipole moment for which causal propagation occurs. It should be noted that, in these discussions, the magnetic dipole interactions considered were, respectively, the types in § 4 with $\alpha_2 = 0$ and $\alpha_1 = 0$.

The equivalence theorems of the present paper are now used in conjunction with the above results to give a discussion of the propagation of the spin one fields considered in earlier sections.

Firstly it is noted that, on account of the results of Shamaly and Capri, the causal propagation of the minimally coupled vector field and the equivalence theorem of § 3, if only the simplest magnetic dipole interactions are considered, a causally propagating Takahashi–Palmer field must have gyromagnetic ratio one.

By exactly the same arguments, the formulation of the Takahashi–Palmer field, given in § 3, is seen to propagate causally. However, the introduction of any anomalous magnetic dipole interaction of the type $\alpha_1 = 0$, leads to acausal propagation.

The minimally coupled spin one equation of Hammer *et al* (1968), as a consequence of its differing from the Takahashi–Palmer theory of § 3 by a simple magnetic dipole interaction, and the results of Shamaly and Capri, has acausally propagating solutions.

By the addition of the magnetic dipole interaction, of type $\alpha_1 = 0$, necessary to give a gyromagnetic ratio of one, this theory may be made causal.

To discuss the antisymmetric tensor field considered by Jenkins (1972a), it is sufficient to remember that it is just the dual of the Takahashi–Palmer field, considered in § 3, whence the above remarks about that field carry through without change.

Thus, although a vector field may be used, in a simple manner, to describe a massive, charged spin one particle, with arbitrary anomalous magnetic dipole moment, in an external electromagnetic field, the spin one field propagating causally (Velo and Zwanziger 1969), this situation does not prevail for any of the spin one fields of Hammer *et al* (1968), Takahashi and Palmer (1970) and Jenkins (1972a). For in the latter cases propagation is acausal for all values of gyromagnetic ratio except unity. However, it should be noted that the equivalence theorem of § 4 allows such a system to be described, with causal propagation, by any of these latter formalisms, though in a far from simple manner.

The situation for the Proca field and its analogue in § 3, now follows immediately from the equivalence theorems of §§ 2 and 3 and the preceding remarks of this section. Thus their only simple anomalous magnetic dipole interactions, which lead to only causal propagation, are the type $\alpha_2 = 0$.

7. Discussion

One consequence of the ambiguity, noted by Bludman and Young (1963), involving the introduction of the minimal electromagnetic interaction into the free spin one Proca Lagrangian, is that the concept of the intrinsic magnetic dipole moment is not well defined in the Proca formalism. More precisely, a massive, charged spin one particle with arbitrary magnetic dipole moment may be described by the minimally coupled Proca theory simply by starting from a free Lagrangian, which differs from the usual free Proca Lagrangian by appropriate divergence terms, involving only $V_\mu(x)$ and its derivatives. In the free Lagrangian these divergence terms evidently play no role; however on the introduction of the minimal electromagnetic interaction, they are no longer divergence terms, and they give rise to magnetic dipole interactions. Since the divergence terms considered involve only $V_\mu(x)$, the argument extends trivially to the vector field formalism, in which the concept of intrinsic magnetic dipole moment is, again, not well defined.

In § 5 a further ambiguity, concerning divergence terms which involve only $\phi_{\alpha\beta}(x)$ and its derivatives, in the introduction of the minimal electromagnetic interaction into the Proca formalism, was noted. Again such terms play no role in the free Lagrangian, whilst, on the introduction of the minimal electromagnetic interaction, they give rise to a second type of magnetic dipole interaction. Since such terms involve only $\phi_{\mu\nu}(x)$, this second type of ambiguity also appears in the antisymmetric tensor theory, where it is related to simple magnetic dipole interactions. Thus, again, the concept of intrinsic magnetic dipole moment for the field $\phi_{\alpha\beta}(x)$ is not well defined. By analogous arguments, it may be easily seen that this situation prevails for all the other spin one fields considered in this paper.

In the light of these ambiguities in the introduction of the minimal electromagnetic interaction into spin one theories, the equivalence theorems, proved in §§ 2, 3 and 4, must be reconsidered. Firstly, the theories of Hammer *et al* (1968) and Takahashi and Palmer (1970), and the formulation of the latter presented in this paper, since, on the introduction of the minimal electromagnetic interaction in the usual way, they only differ by simple magnetic dipole interactions of the same type, (namely, $\alpha_1 = 0$), are, in the light of the ambiguity involving divergence terms depending only on $\phi_{\alpha\beta}(x)$, essentially the same theory.

Having noted that these three theories are essentially the same, the equivalence theorems of §§ 2 and 3 demonstrate that equivalent theories of the minimal interaction of all the spin one fields, considered in this paper, with the electromagnetic field, exist. Further, this equivalence was shown to occur for unit gyromagnetic ratio. In view of the above ambiguities, it is pertinent to ask whether or not this equivalence of all the minimally coupled theories, considered in this paper, is valid for any other value of the gyromagnetic ratio. The answer is no, and is a consequence of the equivalence theorems, in § 4, for anomalous magnetic dipole interactions. The reason is that the two types of simple magnetic dipole interaction, to which the ambiguities are related, are not equivalent.

Thus it is concluded that the minimally coupled spin one theories considered in this paper, and those considered by Bludman and Young (1963) are essentially equivalent. The equivalence being in the sense that there is one, and only one, value of the gyromagnetic ratio such that equivalent minimally coupled theories, in all of these formalisms, can be constructed. Taking one as the normal gyromagnetic ratio, anomalous magnetic dipole interactions may now be introduced into any of the above theories. If they are introduced in the simplest manner, the results of § 4 show that some of the theories considered will be inequivalent, equivalent theories, in general, being related in a complicated manner. This situation also prevails for the case of anomalous electric quadrupole interactions. This contrast between the form of the equivalence theorems for the minimal electromagnetic interaction and the other two interactions is due to the fact that the former is special, in that it takes into account, *a priori*, the contact interactions, which are needed to compensate for the differences in the free particle propagators of corresponding fields in the various theories (Jenkins 1972a, 1972b), whilst the latter two do not. It may be remarked that this result is simply a consequence of the form of the minimal gauge-invariant prescription for introducing the electromagnetic interaction. Any such prescription would lead to similar results.

Finally, on the basis of the discussions of acausal propagation in § 6, it is concluded that the fields of Hammer *et al* (1968), Takahashi and Palmer (1970) and Macfarlane and Tait (1972) (Jenkins 1972a) are not suitable for the description, *in a simple manner*, of a classical massive spin one particle, with anomalous magnetic dipole moment, in an external electromagnetic field.

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